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ON LONGITUDINAL VIBRATIONS OF THE EDGES OF A PLANE CRACK IN AN ELASTIC LAYER *

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The problem of steady longitudinal vibrations of the edges of a plane crack in an elastic layer is considered. The problem is reduced to the solution of an integral equation of the first kind by integral transform methods. The solution of this problem is constructed by using the asymptotic method of "large λ" /1/, as well as the Ritz method. The asymptotic method enabled a solution of the problem to be obtained in a form convenient for practical application. Computational formulas are presented for the intensity coefficient of the tangential stresses that originate outside the crack on its continuation, as well as for the functions characterizing the longitudinal displacements of the crack edges.

1. Formulation of the problem. In the middle plane of an elastic layer of thickness $2h$, let there be a plane crack occupying the domain $y=\pm 0$, $|x|\leqslant a$, $|z|<\infty$. The layer faces are load-free. A load $\tau_{yz}=\pm\tau\cos\omega t$ is applied to the crack edges (t is the time, and the plus and minus signs correspond to the upper and lower edges of the crack). This problem can be reduced /2/ to the solution of the following integral equation by a generalized Fourier transform:

$$\int_{-1}^{1}\varphi(\eta)\,d\eta\int_{\Gamma}\sqrt{u^2-\varkappa^2}\,\mathrm{th}\,(\sqrt{u^2-\varkappa^2})\exp\left(iu\frac{\eta-\xi}{\lambda}\right)du=2\pi\frac{\tau a}{G}\lambda^2 \tag{1.1}$$

$$\varkappa=\omega h\sqrt{\rho/G},\quad \lambda=h/a,\quad \xi=x/a,\quad |\xi|\leqslant 1$$

Here G is the shear mulus, and ρ is the density of the elastic medium. The function $\varphi(\xi)$ is related to the projection W of the displacement vector on the Oz axis $y=\pm 0$, $|x|\leqslant a$ by the formula

$$W=\pm\mathrm{Re}\,\{\varphi(x/a)\exp(i\omega t)\} \tag{1.2}$$

The kernel of the integral equation (1.1) should be understood in the sense of generalized functions. The contour Γ in the kernel coincides with the real axis everywhere outside the neighbourhood of real poles of the function $\mathrm{th}\sqrt{u^2-\varkappa^2}$, and bypasses the positive real poles from above, and the negative poles from below /2/.

The poles of the function $\mathrm{th}\sqrt{u^2-\varkappa^2}$ are determined from the equation $\mathrm{ch}\sqrt{u^2-\varkappa^2}=0$ and have the form

$$u=\pm u_n,\quad u_n=\sqrt{\varkappa^2-\pi^2(n-{}^1/_2)^2}\quad (n=1,\ 2,\ \ldots)$$

i.e., for $\varkappa<\pi/2$ lie on the imaginary axis symmetrically about the real axis. For $\pi/2<\varkappa<\infty$ a finite number of poles of the function $\mathrm{th}\sqrt{u^2-\varkappa^2}$ will be on the real axis.

We superpose the contour Γ in (1.1) on the real axis. We consequently obtain

$$\int_{-1}^{1}\varphi(\eta)\Big\{\int_{-\infty}^{\infty}\sqrt{u^2-\varkappa^2}\exp(ilu)\,\mathrm{th}\sqrt{u^2-\varkappa^2}\,du+ \tag{1.3}$$

$$\left[1+\mathrm{sign}\left(\varkappa-\frac{\pi}{2}\right)\right]\frac{\pi^3}{4}i\sum_{n=1}^{N}\frac{(2n-1)^2}{u_n}\cos(lu_n)\Big\}\,d\eta=2\pi\frac{\tau a}{G}\lambda^2$$

$$l=\frac{\eta-\xi}{\lambda}\left(\varkappa\neq\pi n-\frac{\pi}{2};\ n=1,2,\ldots,N\right),\quad |\xi|\leqslant 1$$

The terms outside the integral in the kernel of (1.3) are due to the presence of poles on the real axis for the function $\mathrm{th}\sqrt{u^2-\varkappa^2}$, and N is the number of positive real poles of this function (cases of double poles at the origin are not examined). Taking account of the value of the integral /3/

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$$\int_{-\infty}^{\infty} \frac{du}{u-c} = 0 \quad (\operatorname{Im} c = 0)$$

we convert (1.3) to a form convenient for the utilization of semi-analytical methods of solving this equation which are realized using a computer

$$\int_{-1}^{1} \varphi(\eta) \Big\{ \int_0^{\infty} [\sqrt{u^2-\varkappa^2} \operatorname{th} \sqrt{u^2-\varkappa^2} \cos(lu) + \tag{1.4}$$

$$\Big[1 + \operatorname{sign}\Big(\varkappa - \frac{\pi}{2}\Big)\Big] \frac{\pi^2}{4} \sum_{n=1}^{N} \frac{(2n-1)^2}{u^2 - u_n^2} \cos(lu_n)\Big] du +$$

$$\Big[1 + \operatorname{sign}\Big(\varkappa - \frac{\pi}{2}\Big)\Big] \frac{\pi^3}{8} i \sum_{n=1}^{N} \frac{(2n-1)^2}{u_n} \cos(lu_n)\Big\} d\eta = \pi \frac{\tau a}{G} \lambda^2$$

2. The method of "large λ". It is more convenient to consider

$$\int_{-1}^{1} \psi(\eta) q\Big(\frac{\eta - \xi}{\lambda}\Big) d\eta = -\pi \frac{\tau a}{G} \lambda, \quad |\xi| \leqslant 1 \tag{2.1}$$

$$q(l) = q_1(l) + q_2(l) \tag{2.2}$$

$$q_1(l) = \int_0^{\infty} \frac{\sqrt{u^2-\varkappa^2}}{u} \sin(lu)\, du$$

$$q_2(l) = \int_0^{\infty} \Big\{ \frac{\sqrt{u^2-\varkappa^2}}{u} (\operatorname{th} \sqrt{u^2-\varkappa^2} - 1) \sin(lu) +$$

$$\Big[1 + \operatorname{sign}\Big(\varkappa - \frac{\pi}{2}\Big)\Big] \frac{\pi^2}{4} \sum_{n=1}^{N} \frac{(2n-1)^2}{u_n (u^2 - u_n^2)} \sin(lu_n)\Big\} du +$$

$$\Big[1 + \operatorname{sign}\Big(\varkappa - \frac{\pi}{2}\Big)\Big] \frac{\pi^3}{8} i \sum_{n=1}^{N} \frac{(2n-1)^2}{u_n^2} \sin(lu_n)$$

for the realization of the method in /1/ to solve (1.4).

The integral equation (2.1) with the kernel (2.2) is equivalent to (1.4) when the following conditions are satisfied:

$$\varphi(\pm 1) = 0, \quad \varphi'(\xi) \equiv \psi(\xi) \tag{2.3}$$

We note that the singularities of the integrand in (2.2) can be eliminated.

In conformity with the limit absorption principle /2/, the function $q_1(l)$ can be represented in the following form ($H_1^{(2)}(z)$ is the Hankel function /4/, and C is the Euler constant):

$$q_1(l) = \frac{\pi}{2} \varkappa i \int \frac{H_1^{(2)}(l)}{l} dl = \frac{1}{l} + \sum_{m=0}^{\infty} \Big(a_m + i \frac{\pi}{2} B_m + B_m \ln|l|\Big) l^{2m+1}$$

$$B_m = \frac{(-1)^m (2m-1)!!\, \varkappa^{2m+2}}{(2m+1)!\,(2m+2)!!}$$

$$a_m = B_m \Big[C - \ln \frac{2}{\varkappa} - \frac{(2m+2)^{-1}}{2m+1} - \sum_{k=1}^{m+1} \frac{1}{k}\Big], \quad m = 0, 1, 2, \ldots$$

The function $q_2(l)$ can be expanded in the series

$$q_2(l) = \sum_{m=0}^{\infty} (c_m + i d_m) l^{2m+1}$$

where c_m and d_m are representable on the basis of the last equation in (2.2) in the form

$$c_m = -i \frac{\pi}{2} B_m + f_m \quad (m = 0, 1, \ldots)$$

$$f_m = \frac{(-1)^m \pi^2}{8(2m+1)!} \Big[1 + \operatorname{sign}\Big(\varkappa - \frac{\pi}{2}\Big)\Big] \sum_{n=1}^{N} (2n-1)^2 u_n^{2m-l} \times$$

$$\ln \frac{\varkappa + u_n}{\varkappa - u_n} - \frac{(-1)^m}{(2m+1)!} \int_0^{\varkappa} \Big\{ \sqrt{\varkappa^2 - u^2} \operatorname{tg}(\sqrt{\varkappa^2 - u^2})\, u^{2m} -$$

$$\left[1+\operatorname{sign}\left(\varkappa-\frac{\pi}{2}\right)\right]\frac{\pi^2}{4}\sum_{n=1}^{N}\frac{(2n-1)^2 u_n^{2m}}{u^2-u_n^{\,2}}\Big\}\,du-$$

$$\frac{(-1)^m}{(2m+1)!}\int_{\varkappa}^{\infty}(1-\operatorname{th}\sqrt{u^2-\varkappa^2})\sqrt{u^2-\varkappa^2}\,u^{2m}\,du$$

$$d_m=\frac{(-1)^m\pi^3}{8(2m+1)!}\left[1+\operatorname{sign}\left(\varkappa-\frac{\pi}{2}\right)\right]\sum_{n=1}^{N}(2n-1)^2u_n^{2m-1}$$

Taking account of the results obtained, we write the series expansion of $q(l)$ in the form (a_m, f_m, d_m are real coefficients)

$$q(l)=\frac{1}{l}+\sum_{m=0}^{\infty}(A_m+B_m\ln|l|)\,l^{2m+1},\quad A_m=a_m+f_m+id_m \tag{2.4}$$

Inserting (2.4) into (2.1), we obtain the following integral equation of the second kind by regularization

$$\psi(\xi)=-\frac{\tau a}{G}\frac{\xi}{\sqrt{1-\xi^2}}+\frac{1}{\pi\sqrt{1-\xi^2}}\sum_{m=0}^{\infty}\frac{1}{\lambda^{2m+2}}\times \tag{2.5}$$
$$\int_{-1}^{1}\frac{\sqrt{1-\eta^2}}{\eta-\xi}\,d\eta\int_{-1}^{1}\psi(\gamma)\left[A_m+B_m\ln\frac{|\gamma-\eta|}{\lambda}\right](\gamma-\eta)^{2m+1}\,d\gamma$$

We seek the solution of (2.5) in the form

$$\psi(\xi)=\frac{\tau a}{G}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\psi_{mn}(\xi)\,\lambda^{-2m}\ln^n\lambda \tag{2.6}$$

Inserting $\psi(\xi)$ in the form of (2.6) into the left and right sides of (2.5) and then equating expressions for identical powers of $\lambda^{-2m}\ln^n\lambda$ we obtain equations determining $\psi_{mn}(\xi)$

$$\psi_{00}(\xi)=-\frac{\xi}{\sqrt{1-\xi^2}} \tag{2.7}$$
$$\psi_{10}(\xi)=\frac{1}{\pi\sqrt{1-\xi^2}}\int_{-1}^{1}\frac{\sqrt{1-\eta^2}}{\eta-\xi}\,d\eta\int_{-1}^{1}\psi_{00}(\gamma)\times[A_0+B_0\ln|\gamma-\eta|](\gamma-\eta)\,d\gamma$$
$$\psi_{11}(\xi)=-\frac{B_0}{\pi\sqrt{1-\xi^2}}\int_{-1}^{1}\frac{\sqrt{1-\eta^2}}{\eta-\xi}\,d\eta\int_{-1}^{1}\psi_{00}(\gamma)(\gamma-\eta)\,d\gamma$$

After having determined the functions $\psi_{mn}(\xi)$ we obtain

$$\psi(\xi)=-\frac{\tau a}{G}\frac{\xi}{\sqrt{1-\xi^2}}\left[1-(D_0+B_0\xi^2)\frac{1}{2\lambda^2}+(F_0+F_1\xi^2+F_2\xi^4)\frac{1}{\lambda^4}+O(\lambda^{-6}\ln^3\lambda)\right] \tag{2.8}$$
$$F_0=\frac{1}{24}B_0^2+\frac{3}{16}B_0D_0+\frac{1}{4}D_0^2+\frac{13}{16}B_1+\frac{3}{8}D_1$$
$$F_1=\frac{5}{48}B_0^2+\frac{1}{4}B_0D_0-\frac{15}{8}B_1-\frac{3}{2}D_1,\quad F_2=\frac{1}{24}B_0^2-\frac{1}{4}B_1$$
$$D_n=A_n-B_n\ln 2\lambda\quad(n=0,\ 1,\ 2,\ \ldots)$$

Taking account of (2.3) the following function can be found from (2.8)

$$\varphi(\xi)=\frac{\tau a}{G}\sqrt{1-\xi^2}\left\{1-\left[D_0+\left(\frac{2}{3}+\frac{1}{3}\xi^2\right)B_0\right]\frac{1}{2\lambda^2}+\right. \tag{2.9}$$
$$\left.+\left[F_0+\left(\frac{2}{3}+\frac{1}{3}\xi^2\right)F_1+\left(\frac{8}{15}+\frac{4}{15}\xi^2+\frac{1}{5}\xi^4\right)F_2\right]\times\frac{1}{\lambda^4}+O(\lambda^{-6}\ln^3\lambda)\right\}$$

Therefore, the function W characterizing the displacement of points of the crack edges can be represented on the basis of (1.2) and (2.9) in the form

$$W|_{y=\pm0,\,|x|\leqslant a}=\pm\frac{\tau}{G}\sqrt{a^2-x^2}\operatorname{Re}\Big\langle\Big\{1- \tag{2.10}$$
$$\left[D_0+\left(\frac{2}{3}+\frac{1}{3}\frac{x^2}{a^2}\right)B_0\right]\frac{1}{2\lambda^2}+\left[F_0+\left(\frac{2}{3}+\frac{1}{3}\frac{x^2}{a^2}\right)F_1+\right.$$
$$\left.\left(\frac{8}{15}+\frac{4}{15}\frac{x^2}{a^2}+\frac{1}{5}\frac{x^4}{a^4}\right)F_2\right]\frac{1}{\lambda^4}+O(\lambda^{-6}\ln^3\lambda)\Big\}e^{i\omega t}\Big\rangle$$

The intensity coefficient of the tangential stresses K_{III} that occur outside the crack on its continuation can be determined from the formulas

$$K_{III} = \lim_{x\to a+0} \sqrt{2\pi(x-a)}\, \tau_{yz}|_{y=+0} = -G \lim_{x\to a-0} \sqrt{2\pi(a-x)}\, W_x'|_{y=+0} \tag{2.11}$$

From (2.10) and (2.11) we obtain

$$K_{III} = \tau\sqrt{a\pi}\, \mathrm{Re}\left\{\left[1-(D_0+B_0)\frac{1}{2\lambda^2}+(F_0+F_1+F_2)\times\frac{1}{\lambda^4}+O(\lambda^{-6}\ln^3\lambda)\right]e^{i\omega t}\right\} \tag{2.12}$$

We note that the solution of the corresponding problem of a crack in space can be obtained from (2.10) and (2.12). To this end, terms corresponding to $q_2(l)$ must be neglected in the solution obtained, while the parameter λ must be replaced by $\varepsilon\varkappa$ in the remaining terms, where $\varepsilon = (a\omega)^{-1}\sqrt{G/\rho}$. For instance, we obtain from (2.12)

$$\begin{aligned}
&K_{III} = \tau\sqrt{a\pi}\,(\Omega_1\cos\omega t+\Omega_2\sin\omega t) \\
&\Omega_1 = 1+\left(0{,}3273+\tfrac{1}{4}\ln\varepsilon\right)\varepsilon^{-2}+\left(-0{,}0554-0{,}1563\ln\varepsilon+\right.\\
&\quad\left.\tfrac{1}{16}\ln^2\varepsilon\right)\varepsilon^{-4}+O(\varepsilon^{-6}\ln^3\varepsilon) \\
&\Omega_2 = 0{,}3927\varepsilon^{-2}+(0{,}2447+0{,}1963\ln\varepsilon)\,\varepsilon^{-4}+O(\varepsilon^{-6}\ln^3\varepsilon)
\end{aligned} \tag{2.13}$$

Formulas (2.13) can be used for $2\leqslant\varepsilon<\infty$. The results of computations using these formulas agree with the results presented in /5/.

3. The Ritz method. When using the Ritz method /6/, the integral equation of the problem under consideration should be taken in the form (1.4). Taking into account the evenness of the function $\varphi(\xi)$, we convert this equation to the form

$$\begin{aligned}
&\int_{-1}^{1}\varphi(\eta)\left\{\int_0^\infty\left[\sqrt{u^2-\varkappa^2}\,\mathrm{th}\sqrt{u^2-\varkappa^2}\cos\frac{\eta u}{\lambda}\cos\frac{\xi u}{\lambda}+\right.\right.\\
&\left[1+\mathrm{sign}\left(\varkappa-\frac{\pi}{2}\right)\right]\frac{\pi^2}{4}\sum_{n=1}^{N}\frac{(2n-1)^2}{u^2-u_n^2}\cos\frac{\eta u_n}{\lambda}\cos\frac{\xi u_n}{\lambda}\Big]\,du+\\
&\left.\left[1+\mathrm{sign}\left(\varkappa-\frac{\pi}{2}\right)\right]\frac{\pi^3}{8}i\sum_{n=1}^{N}\frac{(2n-1)^2}{u_n}\cos\frac{\eta u_n}{\lambda}\cos\frac{\xi u_n}{\lambda}\right\}d\eta=f \\
&(f=\pi\tau aG^{-1}\lambda^2,\quad |\xi|\leqslant 1)
\end{aligned} \tag{3.1}$$

We seek the solution of (3.1) in the form

$$\varphi(\xi)=\frac{\tau a}{G}\sqrt{1-\xi^2}\sum_{m=0}^{M}\frac{(-1)^m}{2m+1}X_mU_{2m}(\xi) \tag{3.2}$$

where $U_{2m}(\xi)$ are Chebyshev polynomials of the second kind, and X_m are coefficients to be determined. Denoting the integral operator acting in (3.1) on the function φ in terms of $\mathbf{L}$, we write this equation in the form

$$\mathbf{L}\varphi = f \tag{3.3}$$

We consider the functional

$$F(\varphi)=(\mathbf{L}\varphi,\varphi)-2(\varphi,f)\quad\left((\varphi_1,\varphi_2)=\int_{-1}^{1}\varphi_1(\xi)\varphi_2(\xi)\,d\xi\right) \tag{3.4}$$

Inserting $\varphi(\xi)$ in the form (3.2) into (3.4) and using the condition for the minimum of the functional $F(\varphi)$, i.e., $\partial F/\partial X_m = 0$ $(m=0,1,\ldots,M)$, we obtain the following system of linear algebraic equations in X_m (δ_{mn} is the Kronecker delta):

$$\sum_{m=0}^{M}X_mR_{mn}=\frac{1}{2}\delta_{n0}\quad(n=0,1,\ldots,M) \tag{3.5}$$

$$\begin{aligned}
&R_{mn}=\frac{\delta_{mn}}{2(2m+1)}+\int_0^\infty\left\{\left[\sqrt{u^2-\varkappa^2}\,\mathrm{th}\sqrt{u^2-\varkappa^2}-u\right]\times\right.\\
&u^{-2}J_{2m+1}\left(\frac{u}{\lambda}\right)J_{2n+1}\left(\frac{u}{\lambda}\right)+\left[1+\mathrm{sign}\left(\varkappa-\frac{\pi}{2}\right)\right]\frac{\pi^2}{4}\times\\
&\left.\sum_{j=1}^{N}u_j^{-2}(u^2-u_j^2)^{-1}(2j-1)^2J_{2m+1}\left(\frac{u_j}{\lambda}\right)J_{2n+1}\left(\frac{u_j}{\lambda}\right)\right\}du+
\end{aligned} \tag{3.6}$$

$$\left[1+\operatorname{sign}\left(\varkappa-\frac{\pi}{2}\right)\right]\frac{\pi^3}{8}\,i\sum_{j=1}^{N}(2j-1)^2 u_j^{-3} J_{2m+1}\left(\frac{u_j}{\lambda}\right) J_{2n+1}\left(\frac{u_j}{\lambda}\right)$$

Formulas (7.324,2), (6.574,2) in /4/ are used to obtain expressions (3.6) determining R_{mn}. The coefficients R_{mn} should be evaluated taking the symmetry $R_{mn} = R_{nm}$ into account.

A system of functions $\varphi_m(\xi) = \sqrt{1-\xi^2}U_{2m}(\xi)$, that are elements of a real Hilbert space, it selected as the system of coordinate functions for realization of the Ritz method of solving the integral equation (3.1). The coefficients X_m in the linear combination of these functions (3.2) are real for $\lambda < \infty$ and $\varkappa < \pi/2$ and complex for $\varkappa > \pi/2$. This deduction follows from the representation (3.6) for the coefficients R_{mn}. It therefore follows that there is no phase shift in the vibrations of the load τ_{yz} applied to the crack edges and in the vibrations of the points of the crack edges for $\lambda < \infty$ and $\varkappa < \pi/2$. In the remaining cases, including the case of a crack in space, a phase shift will occur in the vibrations mentioned.

We also note that (3.5) and (3.6) can be obtained by the method of orthogonal polynomials. It is here necessary to use the series expansion of $\cos(Az)$ in Chebyshev polynomials of the second kind $U_{2n}(z)$

$$\cos(Az) = \frac{2}{A}\sum_{n=0}^{\infty}(-1)^n(2n+1)J_{2n+1}(A)U_{2n}(z) \tag{3.7}$$

The validity of (3.7) can easily be established by using (7.324,2) in /4/ and the orthogonality condition for the polynomials $U_{2n}(z)$ /4/. Expanding the solution of (3.1) in the form

$$\varphi(\xi) = \frac{\tau a}{G}\sqrt{1-\xi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}X_m U_{2m}(\xi) \tag{3.8}$$

and applying the procedure of the method of orthogonal polynomials by using (3.7), we obtain the following infinite linear algebraic system to determine the coefficients X_m of expansion (3.8):

$$\sum_{m=0}^{\infty}X_m R_{mn} = \frac{1}{2}\delta_{n0} \quad (n = 0, 1, 2, \ldots) \tag{3.9}$$

where the R_{mn} are given by (3.6). Applying the method of reduction to (3.9), we arrive at the final system (3.5).

4. **Comparison of the results.** We obtain as a result of investigating the problem by the Ritz method taking the value $U_{2m}(1)/(2m+1) = 1$ into account:

$$K_{\mathrm{III}} = \tau\sqrt{a\pi}\sum_{m=0}^{M}(-1)^m \operatorname{Re}(X_m e^{i\omega t}) \tag{4.1}$$

We represent the result of solving the problem by the method of large λ in the form

$$K_{\mathrm{III}} = \tau\sqrt{a\pi}\operatorname{Re}[(N_1 + iN_2)e^{i\omega t}] \tag{4.2}$$

The amplitude values of the tangential stress intensity coefficient are respectively calculated for each of the solutions obtained, from the formulas

$$\max_t K_{\mathrm{III}} = \tau\sqrt{a\pi}\left\{\left[\sum_{m=0}^{M}(-1)^m \operatorname{Re} X_m\right]^2 + \left[\sum_{m=0}^{M}(-1)^m \operatorname{Im} X_m\right]^2\right\}^{1/2} \tag{4.3}$$

$$\max_t K_{\mathrm{III}} = \tau\sqrt{a\pi}\{N_1^2 + N_2^2\}^{1/2} \tag{4.4}$$

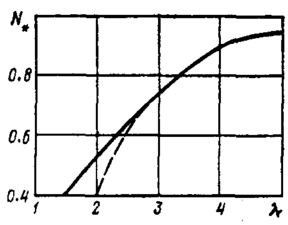

Fig.1

As an illustration we show in the figure the results of calculating the parameter $N_* = \max_t K_{\mathrm{III}}/(\tau\sqrt{a\pi})$ by (4.3) (the solid line) and (4.4) (the dashed line) by each of the methods considered for $u_1 = 1$.

The accuracy of the investigation of the problem by the Ritz method is established by comparing the results for a different number of coordinate functions. In particular, for $0.5 \leqslant \lambda < \infty$ the results of the calculations for $M = 4$ and $M = 8$ differ by not more than 1%. Therefore, in the case under consideration, the solution of the problem by the method of large λ yields quite exact results for $\lambda \geqslant 3$.

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ON THE LAGRANGE PROBLEM OF THE MEAN MOTION OF PERIHELIA*

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It is shown that themean motion of perihelia in the Lagrange sense in a non-resonance set is uniformly continuous in the initial phases of the frequency function.

The dynamics of a planetary system such as the solar system is examined. To a first approximation of perturbation theory, when the squares of the orbit eccentricities can be neglected compared with the eccentricities themselves, the evolution of the Laplace vector is described by the function

$$A(t) = \sum_{m=0}^{n} a_m \exp[2\pi i(\lambda_m t + \varphi_{0m})]$$

where the constants $a_m, \lambda_m, \varphi_{0m}$ are determined in terms of the planet mass and the initial conditions. The mean motion of the perihelion is defined as

$$\mu(\mathbf{a}, \boldsymbol{\lambda}, \varphi_0) = \lim_{t\to\infty} \frac{1}{t} \arg A(t)$$

Lagrange showed that $\mu = 2\pi\lambda_0$ if $a_0 \geqslant a_1 + a_2 + \ldots + a_n$. In the non-trivial case when this condition is not satisfied, the mean motion is calculated for $n = 2$ /1/ and for arbitrary n /2/ for the non-resonance set of frequencies $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_n)$: $< k, \lambda > \neq 0$, $\forall \mathbf{k} \in Z^{n+1}$, $\mathbf{k} \neq 0$ and has the form

$$\mu(\mathbf{a}, \boldsymbol{\lambda}) = 2\pi \sum_{m=0}^{n} \lambda_m w^m(\mathbf{a})$$

$$\sum_{m=0}^{n} w^m(\mathbf{a}) = 1, \quad w^m(\mathbf{a}) \geqslant 0, \quad \mathbf{a} = (a_0, \ldots, a_n)$$

The existence of mean motion for an arbitrary set of frequencies $\boldsymbol{\lambda}$ is proved in /3/.

Let $\mathbf{a}, \boldsymbol{\lambda}$ be certain continuous functions of the parameter $\alpha \in R^s$.

Assertion. If for $\alpha = \alpha_0$ the vector $\boldsymbol{\lambda}(\alpha_0)$ is non-resonant, then uniformly in $\varphi_0 \in T^{n+1}$ $\{\varphi \bmod 1\}$

$$\lim_{\alpha\to\alpha_0} \mu(\mathbf{a}(\alpha), \boldsymbol{\lambda}(\alpha), \varphi_0) = \mu(\mathbf{a}(\alpha_0), \boldsymbol{\lambda}(\alpha_0))$$

Remarks. 1°. For fixed α the function $\mu(\mathbf{a}, \boldsymbol{\lambda}, \varphi_0)$ is generally discontinuous in φ_0, when $\boldsymbol{\lambda}(\alpha)$ is a resonance vector.

2°. If the function $A(t)$ vanishes for certain values of the time, its argument is not defined. In such a case it is customary to distinguish the "first" and "left" arguments of the function $A(t)$. When passing through a zero of multiplicity p the right argument of the function $A(t)$ receives an increment πp as $t\to\infty$, while the left receives the increment $(-\pi p)$ /4/. The right and left mean motions μ^+ and μ^- /3/ that are in agreement for non-resonance

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